Chapter 2

Optimal Control

Optimal control is the standard method for solving dynamic optimization problems, when those problems are expressed in continuous time. It was developed by *inter alia* a bunch of Russian mathematicians among whom the central character was Pontryagin. American economists, Dorfman (1969) in particular, emphasized the economic applications of optimal control right from the start.

1. An Economic Interpretation of Optimal Control Theory

This section is based on Dorfman's (1969) excellent article of the same title. The purpose of the article was to derive the *technique* for solving optimal control problems by thinking through the *economics* of a particular problem. In so doing, we get a lot of intuition about the economic meaning of the solution technique.

The Problem

- A firm wishes to maximize its total profits over some period of time.
- At any date t, it will have inherited a capital stock from its past behavior. Call this stock $k(t)$.
- Given $k(t)$, the firm is free to make a decision, $x(t)$, which might concern output, price, or whatever.
- Given $k(t)$ and $x(t)$, the firm derives a flow of benefits per unit of time. Denote this flow by $u(k(t),x(t),t)$.

Consider the situation at a certain time, say t, and let $W\big(k(t),\vec{x},t\big)$ denote the total flow of profits over the time interval $[t,T]$. That is

$$W\big(k(t),\vec{x},t\big) = \int_t^T u\big(k(\tau),x(\tau),\tau\big)d\tau \,,$$

where $\vec{x}$ is not an ordinary number, but the entire time path of the decision variable x, from date t through to date T.

The firm is free to choose $x(t)$ at each point in time, but it cannot independently choose the capital stock, $k(t)$, at each point in time. The time path of $k(t)$ will depend on past values of k and on the decision, $x(t)$, that the firm makes. Symbolically, we write this constraint in terms of the time-derivative for $k(t)$:

$$\dot{k}(t) = f\big(k(t),x(t),t\big)\,.$$

The decision made at time t has two effects:

- it affects the flow of profits at t, $u\big(k(t),x(t),t\big)\,.$

- it affects the value of k, which in turn will affect *future* profits.

The essence of the optimal control problem is this: Choose the entire time path, $\vec{x}$, to maximize $W\big(k(t),\vec{x},t\big)$, subject to the equation of motion governing $k(t)$.

This is a very difficult problem (as Dorfman says, "not only for beginners"), because standard calculus techniques tell us how to choose an **optimal value** for a variable, not an **optimal time path**. The strategy for solving this type of problem is to transform it into one which demands we find only a single number (or a few numbers). This is something we know how to do with ordinary calculus. There are various ways to make this transformation. Optimal control theory is the most straightforward *and* the most general.

The Firm's Capital Problem

Given

$$W\big(k(t),\vec{x},t\big) = \int_t^T u\big(k(\tau),x(\tau),\tau\big)d\tau \,,$$

we break this into two parts:

- A short interval of length Δ, beginning at time t,

- The remainder, from $t+\Delta$ to T.

We assume that Δ is so short that the firm would not change $x(t)$ in the course of the interval, even if it could. We will subsequently let Δ shrink to zero in the usual manner of deriving results in calculus, so that our assumption must be true in the limit. Thus, we can write

$$W\big(k(t),\vec{x},t\big) = u\big(k,x(t),t\big)\Delta + \int_{t+\Delta}^{T} u\big(k(\tau),x(\tau)\big),\tau d\tau$$

Contribution to W over the interval Δ, where $x(t)$ is constant.

Starting capital stock is $k(t+\Delta)$, which has changed from $k(t)$ in a manner influenced by $x(t)$.

As the integral $\int_{t+\Delta}^{T} \bullet d\tau$ has exactly the same form as $\int_{t}^{T} \bullet d\tau$, with only the lower limit changed, we can use our definition of W to write

$$W\big(k(t),\vec{x},t\big) = u\big(k(t),x(t),t\big)\Delta + W\big(k(t+\Delta),\vec{x},t+\Delta\big).$$

Let $\vec{x}\,^*$ denote the time path of x that maximizes W, and let V^* denote that maximum:

$$V\,^*\big(k(t),t\big) = \max_{\vec{x}} \; W\big(k(t),\vec{x},t\big).$$

We now consider a peculiar policy. Over the interval Δ, choose an arbitrary $x(t)$. But from $t+\Delta$, choose the optimal path, $\vec{x}\,^*$. The payoff from this policy is

$$V\big(k(t),x(t),t\big) = u\big(k(t),x(t),t\big)\Delta + V\,^*\big(k(t+\Delta),t+\Delta\big)$$

Benefits accruing from decision $x(t)$ in interval Δ.

Maximum benefits obtainable over the interval $[t+\Delta,T]$, given that $k(t+\Delta)$ is determined by $k(t)$ and $x(t)$.

$x(t)$ has disappeared from the argument list, because it has already been optimally chosen in defining V^*.

This is now a problem of ordinary calculus: find the **value** of $x(t)$ that maximizes $V\big(k(t), x(t), t\big)$. Take the derivative of the last equations with respect to $x(t)$ and set to zero in the usual manner:

$$\frac{\partial V}{\partial x(t)} = \Delta \frac{\partial u}{\partial x(t)} + \frac{\partial V^*}{\partial k(t+\Delta)} \cdot \frac{\partial k(t+\Delta)}{\partial x(t)} = 0 \, .$$

Now:

- Because Δ is small,

$$k(t+\Delta) \approx k(t) + \Delta \dot{k}(t)$$
$$= k(t) + \Delta f\big(k(t), x(t), t\big),$$

and thus

$$\frac{\partial k(t+\Delta)}{\partial x(t)} = \Delta \frac{\partial f\big(k(t), x(t), t\big)}{\partial x(t)} \, .$$

- $\partial V^*\big(k(t+\Delta), t+\Delta\big)/\partial k(t+\Delta)$ is the increment to V^* resulting from a marginal increase in $k(t{+}\Delta)$. That is, it is the **marginal value of capital**. Let $\lambda(t{+}\Delta)$ denote the marginal value.

Then, we get the following:

$$\Delta \frac{\partial u}{\partial x(t)} + \lambda(t+\Delta)\Delta \frac{\partial f}{\partial x(t)} = 0 \, .$$

Canceling the Δs,

$$\frac{\partial u}{\partial x(t)} + \lambda(t+\Delta)\frac{\partial f}{\partial x(t)} = 0 \, ,$$

and letting $\Delta \to 0$ yields

$$\boxed{\frac{\partial u\big(k(t), x(t), t\big)}{\partial x(t)} + \lambda(t)\frac{\partial f\big(k(t), x(t), t\big)}{\partial x(t)} = 0 \, .} \tag{C.1}$$

This is the first of three necessary conditions we need to solve any optimal control problem. Let us interpret this equation. Along the optimal path, the marginal short-run

effect of a change in decision (i.e. $\partial u / \partial x$) must exactly counterbalance the effect on total value an instant later. The term $\lambda \partial f / \partial x$ is the effect of a change in x on the capital stock, multiplied by the marginal value of the capital stock. Put another way, $x(t)$ should be chosen so that the marginal immediate gain just equals the marginal long-run cost.

Condition C.1 gives us the condition that must be satisfied by the choice variable $x(t)$. However, it includes the variable $\lambda(t)$, the marginal value of capital, and we do not yet know the value of this variable. The next task, then is to characterize $\lambda(t)$. Now, suppose $x(t)$ is chosen so as to satisfy C.1. Then, we will have

$$V*\big(k(t),t\big) = u\big(k(t),x(t),t\big)\Delta + V*\big(k(t+\Delta),t+\Delta\big).$$

Recall that $\lambda(t)$ is the change in the value of the firm given a small change in the capital stock. To find an expression for this, we can differentiate the value function above with respect to $k(t)$:

$$\frac{\partial V*\big(k(t),t\big)}{\partial k(t)} = \lambda(t) = \Delta\frac{\partial u}{\partial k(t)} + \lambda(t+\Delta)\frac{\partial k(t+\Delta)}{\partial k(t)}$$

$$= \Delta\frac{\partial u}{\partial k(t)} + \left[\frac{\partial\big[k(t)+\dot{k}(t)\Delta\big]}{\partial k(t)}\right]\big(\lambda(t)+\dot{\lambda}(t)\Delta\big)$$

As Δ is small,
$$\lambda(t+\Delta) \approx \lambda(t) + \dot{\lambda}(t)\Delta$$

$$= \Delta\frac{\partial u}{\partial k(t)} + \left[1 + \Delta\frac{\partial f}{\partial k(t)}\right]\big(\lambda(t)+\dot{\lambda}(t)\Delta\big).$$

Hence,

$$\cancel{\lambda(t)} = \Delta\frac{\partial u}{\partial k(t)} + \cancel{\lambda(t)} + \dot{\lambda}(t)\Delta + \lambda(t)\Delta\frac{\partial f}{\partial k(t)} + \dot{\lambda}(t)\frac{\partial f}{\partial k(t)}\Delta^2$$

Divide through by Δ and let $\Delta \to 0$. The last term vanishes, allowing us to write

$$-\dot{\lambda}(t) = \frac{\partial u\big(k(t),x(t),t\big)}{\partial k(t)} + \lambda(t)\frac{\partial f\big(k(t),x(t),t\big)}{\partial k(t)}. \tag{C.2}$$

This is the second of three necessary conditions we need to solve any optimal control problem. The term $\dot{\lambda}(t)$ is the appreciation in the marginal value of capital, so $-\dot{\lambda}(t)$ is

its depreciation. Condition C.2 states that the depreciation of capital is equal to the sum of its contribution to profits in the immediate interval dt, and its contribution to increasing the value of the capital stock at the end of the interval dt.

Now we have an expression determining the value of the choice variable, $x(t)$, and an expression determining the marginal value of capital, $\lambda(t)$. But both of these depend on the capital stock, $k(t)$, so we need an expression to characterize the behavior of $k(t)$. This is easy to come by, because we already have the resource constraint:

$$\dot{k}(t) = f\big(k(t), x(t), t\big), \tag{C.3}$$

which is the third of our necessary conditions. Note that our necessary conditions include two nonlinear differential equations. As you can imagine, completing the solution to this problem will rely on all the tools developed in the module on differential equations. We'll hold off from that task from the moment.

2. The Hamiltonian and the Maximum Principle

Conditions (C.1) through (C.3) form the core of the so-called Pontryagin Maximum Principle of optimal control. Fortunately, you don't have to derive them from first principles for every problem. Instead, we construct a way of writing down the optimal control problem in standard form, and we use this standard form to generate the necessary conditions automatically.

We can write the firm's problem,

$$\max_{x(t)} \quad \int_t^T u\big(k(\tau), x(\tau), \tau\big)d\tau$$

subject to

$$\dot{k}(t) = f\big(k(t), x(t), t\big),$$

in a convenient way by writing the auxiliary, or **Hamiltonian**, function:

$$H = u\big(k(t), x(t), t\big) + \lambda(t) f\big(k(t), x(t), t\big),$$

and then writing down the necessary conditions:

$$\frac{\partial H}{\partial x} = 0 \tag{C.1}$$

$$\frac{\partial H}{\partial k} = -\dot{\lambda} \tag{C.2}$$

$$\frac{\partial H}{\partial \lambda} = \dot{k} \tag{C.3}$$

The most useful tactic is to *memorize* the form of the Hamiltonian and the three necessary conditions for ready application, while *understanding* the economic meaning of the resulting expressions from the specific example we have worked through. We have expressed the optimal control problem in terms of a specific example: choose investment, $x(t)$, to maximize profits by affecting the evolution of capital, $k(t)$, which has a marginal value, $\lambda(t)$. In general, we choose a **control variable**, $x(t)$, to maximize an objective function by affecting a **state variable**, $k(t)$. The variable $\lambda(t)$ is generically termed the **costate variable** and it always has the interpretation of the **shadow value** of the state variable.

Although the Hamiltonian is most often thought of as a device to remember the optimality conditions, it does have an interpretation related to the rate of accumulation of total value of the firm. Consider a *modified* Hamiltonian of the form,

$$H^* = u\big(k(t), x(t), t\big) + \lambda(t) f\big(k(t), x(t), t\big) + \dot{\lambda}(t) k(t),$$

which can be written as

$$H^* = u\big(k(t), x(t), t\big) + \lambda(t)\dot{k}(t) + \dot{\lambda}(t)k(t). \tag{2.1}$$

This modified Hamiltonian consists of three terms that together represent the accumulation of value to the firm. The first is the flow rate of earnings. The second is the rate of change of the stock variable, multiplied by its shadow price, and the third is the change in the value of the current stock (i.e. the capital gains). Maximizing the modified Hamiltonian, H^*, at every point in time is therefore equivalent to maximizing the flow rate of accumulation of value at every point in time. If the firm were free to choose both k and x simultaneously, it would simply choose values satisfying the first-order conditions

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0\,,$$

and

$$\frac{\partial u}{\partial k} + \lambda \frac{\partial f}{\partial k} + \dot{\lambda} = 0\,.$$

These are just the two necessary conditions already derived. Of course, the decision maker cannot directly choose k, because it is not a choice variable. However, the optimality conditions imply that the firm should choose the time paths of x and λ so that the values of k are the ones the firm would choose to maximize the increment to firm value at each point in time.

It is also easy to see that the integral of the modified Hamiltonian from current time t to the terminal time T represents the firm value remaining to be accumulated. Integrate (2.1):

$$\int_t^T H(k(s), x(s), s) * ds = \int_t^T u\big(k(s), x(s), s\big)\, ds + \int_t^T \lambda(s)\dot{k}(s)ds + \int_t^T \dot{\lambda}(s)k(s)ds$$

$$= \int_t^T u\big(k(s), x(s), s\big)\, ds + \int_t^T \lambda(s)\dot{k}(s)ds + \lambda(s)k(s)\big|_t^T - \int_t^T \lambda(s)\dot{k}(s)ds$$

$$= \int_t^T u\big(k(s), x(s), s\big)\, ds + \big(\lambda(T)k(T) - \lambda(t)k(t)\big)\,. \qquad (2.2)$$

The last equation consists of all the future profits to be earned, plus the change in the value of the stock owned between t and T. The sum of the terms must equal the value of solving the problem between t and T.

The three formulae (C.1) through (C.3) jointly determine the time paths of x, k, and λ. Of course, two of them are differential equations that require us to specify boundary conditions to identify a unique solution. The economics of the problem will usually lead us to particular boundary conditions. For example, at time zero we may have an initial capital stock $k(0)$ that is exogenously given. Alternatively, we may require that at the end of the planning horizon, T, we are left with a certain amount $k(T)$. In the particular economic problem we have studied, a natural boundary condition occurs at T. Given that

the maximization problem stops at T, having any left over capital at time T has no value at all. Thus, the marginal value of k at time T must be zero., That is, we have a boundary condition $\lambda(T){=}0$. The condition $\lambda(T){=}0$ in the capital problem is known as a **transversality condition**. Note that applying this boundary problem to (2.2) leaves us with the original objective function for the capital problem.

There are various types of transversality conditions, and which one is appropriate depends on the economics of the problem. After working through a simple optimal control example, we will study transversality conditions in more detail.

EXAMPLE 2.1

$$\max_{u(t)} \int_0^1 \big(x(t) + u(t)\big) dt \, ,$$

subject to

$$\dot{x}(t) = 1 - u(t)^2 \, ,$$

$$x(0) = 1 \, .$$

Here $u(t)$ is the control and $x(t)$ is the state variable. I have changed the notation from the previous example specifically to keep you on your toes – there is no standard notation for the control, state, and costate variables. There is no particular economic application for this problem.

The first step is to form the Hamiltonian:

$$H = x(t) + u(t) + \lambda(t)\big(1 - u(t)^2\big).$$

We then apply the rules for our necessary conditions:

$$\frac{\partial H}{\partial u} = 1 - 2\lambda(t)u(t) = 0 \, , \tag{2.3}$$

$$\frac{\partial H}{\partial x} = 1 = -\dot{\lambda}(t) \, , \tag{2.4}$$

$$\frac{\partial H}{\partial \lambda} = 1 - u(t)^2 = \dot{x}(t) \, . \tag{2.5}$$

The most common tactic to solve optimal control problems is to manipulate the necessary conditions to remove the costate variable, $\lambda(t)$, from the equations. We can then express the solution in terms of observable variables.

By definition,

$$\lambda(t) = \lambda(0) + \int_0^t \dot{\lambda}(\tau)d\tau \ ,$$

From (2.4) we have $\dot{\lambda}(t) = -1$, so

$$\lambda(t) = \lambda(0) + \int_0^t -1\,d\tau$$

$$= \lambda(0) - t \ .$$

Now, at $t=1$, we know that $\lambda(1)=0$, because having any x left over at $t=1$ does not add value to the objective function. Thus, $\lambda(1) = \lambda(0) - 1 = 0$, implying that $\lambda(0)=1$. Hence,

$$\lambda(t) = 1 - t \ . \tag{2.6}$$

Substitute (2.6) into (2.3) to eliminate the costate variable

$$1 - 2(1 - t)u(t) = 0$$

which solves for

$$u(t) = \frac{1}{2(1 - t)} \tag{2.7}$$

Equation (2.7) is the solution to our problem: it defines the optimal value of the control variable at each point in time.

We can also obtain the value of the state variable from (2.5):

$$\dot{x}(t) = 1 - u(t)^2$$

$$= 1 - \frac{1}{4(1 - t)^2} \ .$$

Integrating gives

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)d\tau$$

$$= 1 + \int_0^t 1 - \frac{1}{4(1-\tau)^2} \, d\tau$$

$$= 1 + t - \frac{1}{4(1-t)} + \frac{1}{4}$$

$$= t - \frac{1}{4(1-t)} + \frac{5}{4}.$$

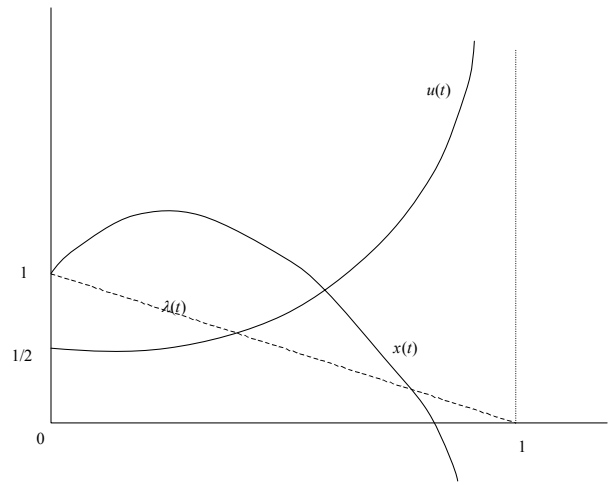Graphically, the solution is as depicted in Figure 2.1. •

FIGURE 2.1

   This is about as simple as optimal control problems get. Of course, nonlinear neces-
sary conditions are much more common in economic problems, and they are much more
difficult to deal with. We will have plenty of practice with more challenging problems in
the remainder of this chapter.

3. Alternative Problem Types and Transversality Conditions

In Dorfman's analysis of the capital problem, it was noted that at the terminal time T, the shadow price of capital, $\lambda(T)$, must be zero. In Example 2.1, we needed to use the condition $\lambda(1)=0$ to solve the problem. These conditions are types of transversality conditions – they tell us how the solution must behave as it crosses the terminal time T. There are various types of conditions one might wish to impose, and each of them implies a different transversality condition. We summarize the possibilities here.
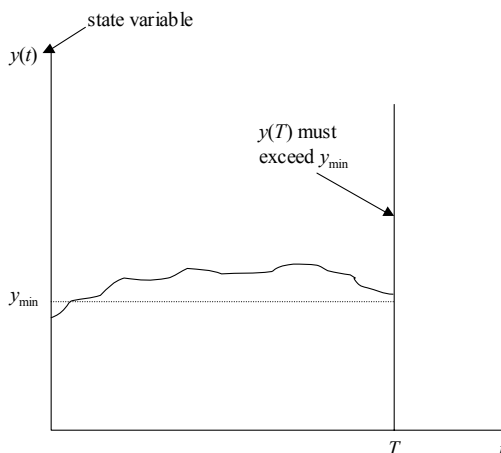
1. Free end point
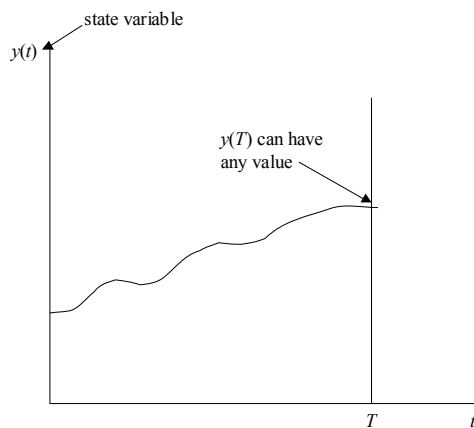
$\lambda(T)=0$.

The shadow value of $y(T)$ is zero. Because the value function ends at T, an increase in $y(T)$ cannot affect the payoff.

state variable

$y(t)$

$y(T)$ can have
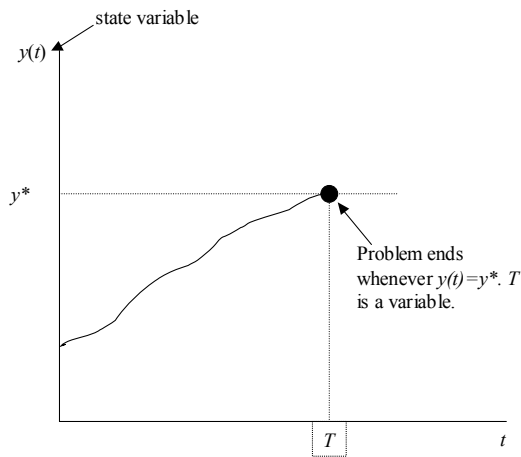any value

T t

2. Constrained end point

$\lambda(T)[y(T)-y_{\min}]=0$.

If $y(T)>y_{\min}$ optimally, then the condition for a free end-point applies: $\lambda(T)=0$. If $y(T)>y_{\min}$ is not optimal, then $\lambda(T)=0$ no longer applies. Instead, we have $y(T)-y_{\min}=0$. The transversality condition for the constrained end-point is the Kuhn-

state variable

$y(t)$

$y(T)$ must
exceed $y_{\min}$

$y_{\min}$

T t

Tucker complementary slackness condition. If, as is common, we have a non-negativity constraint (i.e. $y_{min}=0$), then the transversality condition can be written as $\lambda(T)y(T)=0$. This is a very common transversality condition for economic problems. The same condition also applies for maximum constraints; simply replace y_{min} with y_{max}.

3. T is free but $y(T)$ is fixed.

state variable

$y(t)$

y^*

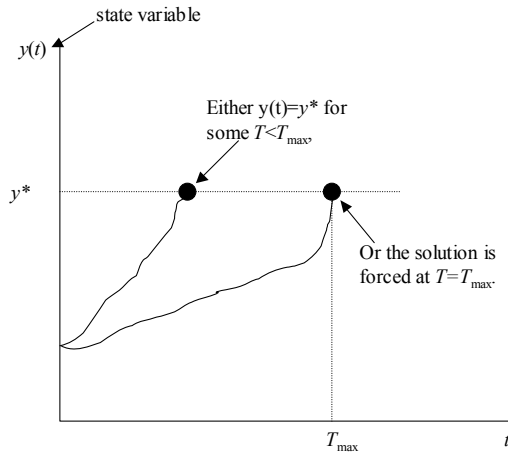Problem ends whenever $y(t)=y^*$. T is a variable.

T

t

$H(T)=0$.

The Hamiltonian,

$$H(t) = f(x,y,t) + \lambda(t)g(x,y,t),$$

equals zero at time T. This sort of transversality condition applies in such problems as devising the shortest path to a particular point. Consider, for example, a problem of building a road up a hill, subject to the constraint that the gradient cannot exceed $z\%$ at any point. The state variable, y is the height of the road (how much of the elevation has been scaled), while T is the length of the road. The choice variable x is the direction the road takes at each point. The problem then is to choose the path of the road, x, so as to minimize the distance T required to get to an elevation of y^* subject to the gradient constraint. These sorts of problems can be quite tricky to solve. The intuition behind the condition $H(T)=0$ is that extending the time horizon beyond T has no value because the job has been completed. That is, there is no more value to accumulate. At the same time, $H(T)$ cannot be negative – if that were the case, it would have been optimal to finish the problem earlier.

4. $y(T)$ is fixed, and $T < T_{\max}$.

state variable

$y(t)$

Either y(t)=y* for some $T < T_{\max}$.

y^*

Or the solution is forced at $T = T_{\max}$.

$T_{\max}$ t

$(T - T_{\max})H(T) = 0$. This is, again a Kuhn-Tucker condition. If $T < T_{\max}$, then $H(T) = 0$, as in 3 above. Otherwise $(T - T_{\max}) = 0$. Imagine adding to the road building example in 3 above the condition that the road cannot be longer than $T_{\max}$. This would happen if there was only enough material to build a road of a certain length. Then, the problem might read as follows: choose the path of the road, x, so as to minimize the distance T required to get to an elevation of y^* subject to the gradient constraint, if this can be done with a total road length less than $T_{\max}$. If it cannot be done, violate the gradient constraint as little as possible to keep the road length at $T_{\max}$.

EXERCISE 3.1. *Solve the following problems, paying particular attention to the appropriate transversality conditions.*

(a) *Fixed end points*

$$\max_u \int_0^1 -u(t)^2 \, dt \,, \quad \text{subject to} \quad \dot{y}(t) = y(t) + u(t), \quad y(0){=}1, \quad y(1){=}0.$$

(b) *Constrained end point*

$$\max_u \int_0^1 -u(t)^2 \, dt \,, \quad \text{subject to} \quad \dot{y}(t) = y(t) + u(t)\,, \quad y(0){=}1, \quad \text{y}(1) \geq 2\,.$$

(c) *T free, but $y(T)$ fixed*

$$\max_u \int_0^T -1 dt \,, \quad \text{subject to} \quad \dot{y}(t) = y(t) + u(t), \quad y(0){=}5, \quad y(T){=}11, \quad u \in [-1,1]$$

EXERCISE 3.2. *Characterize the solution to the following investment problem:*

$$\max_{c} \int_{o}^{T} e^{-\rho t} U(c(t)) dt$$

subject to $\dot{k}(t) = rk(t) - c(t)$, $k(0)=k_0>0$, $k(T) \geq 0$, *and where*
$\lim_{c \to 0} u'(c) = \infty$, *and* $u''(c) < 0$.

4. Multiple Controls and State Variables

The maximum principle of optimal control readily extends to more than one control variable and more than one state variable. One writes the problem with a constraint for each state variable, and an associated costate variable for each constraint. There will then be a necessary condition $\partial H / \partial u_i = 0$ for each control variable, a condition $\partial H / \partial y_i = -\dot{\lambda}_i$ for each costate variable, and a resource constraint for each state variable. This is best illustrated by an example.

EXAMPLE 4.1 (*Fixed endpoints with two state and two control variables*). Capital, $K(t)$ and an extractive resource, $R(t)$ are used to produce a good, $Q(t)$ according to the production function $Q(t) = AK(t)^{1-\alpha} R(t)^{\alpha}$, $0<\alpha<1$. The product may be consumed at the rate $C(t)$, yielding utility $\ln C(t)$, or it may be converted into capital. Capital does not depreciate and the fixed resource at time 0 is X_0. We wish to maximize utility over a fixed horizon subject to endpoint conditions $K(0)=K_0$, $K(T)=0$, and $X(T)=0$. Show that, along the optimal path, the ratio $R(t)/K(t)$ is declining, and the capital output ratio is increasing.

This problem has two state variables, $K(t)$ and $X(t)$, and two control variables, $R(t)$ and $C(t)$. Formally, it can be written as

$$\max_{C,R} \int_{0}^{T} \ln C(t) dt$$

subject to

$$\dot{X}(t) = -R(t), \quad X(0)=X_0, \quad X(T)=0,$$

$$\dot{K}(t) = AK(t)^{1-\alpha}R(t)^\alpha - C(t), \quad K(0)=K_0, \quad K(T)=0.$$

The Hamiltonian is

$$H(t) = \ln C(t) - \lambda(t)R(t) + \mu(t)\Big(AK(t)^{1-\alpha}R(t)^\alpha - C(t)\Big),$$

with necessary conditions

$$\frac{\partial H}{\partial C(t)} = \frac{1}{C(t)} - \mu(t) = 0, \tag{4.1}$$

$$\frac{\partial H}{\partial R(t)} = -\lambda(t) + \alpha\mu(t)AK(t)^{1-\alpha}R(t)^{\alpha-1} = 0, \tag{4.2}$$

$$\frac{\partial H}{\partial X(t)} = 0 = -\dot{\lambda}(t) \tag{4.3}$$

$$\frac{\partial H}{\partial K(t)} = (1-\alpha)\mu(t)AK^{-\alpha}R(t)^\alpha = -\dot{\mu}(t), \tag{4.4}$$

plus the resource constraints. From (4.3), we see that $\lambda(t)$ is constant. Substitute $y(t)=R(t)/K(t)$ into (4.2) and differentiate with respect to time, using the fact that $\lambda(t)$ is constant:

$$(1-\alpha)\frac{\dot{y}(t)}{y(t)} = \frac{\dot{\mu}(t)}{\mu(t)}, \tag{4.5}$$

and using (4.5) in (4.4), we get

$$-y^{-(1+\alpha)}\dot{y}(t) = A. \tag{4.6}$$

This is a nonlinear equation, but it is easy to solve. Note that if you differentiate $y(t)^{-\alpha}/\alpha$ with respect to time, you get the left hand side of (4.6). Hence, if we integrate both sides of (4.6) with respect to t, we get

$$\frac{y(t)^{-\alpha}}{\alpha} + c = At,$$

where c is the constant of integration. Some rearrangement gives

$$Ay(t)^\alpha = \frac{1}{k + \alpha t} \tag{4.7}$$

where $k = \alpha c / A$ is a constant to be determined. Equation (4.7) tells us that y, the ratio of R to K, declines over time. Finally, since

$$\frac{K(t)}{Q(t)} = \frac{K(t)}{AK(t)^{1-\alpha}R(t)^{\alpha}} = \frac{1}{Ay(t)^{\alpha}} = k + \alpha t\,,$$

the capital output ratio grows linearly at the rate α.   •

5. When are Necessary Conditions Also Sufficient?

When are the necessary conditions also sufficient conditions? Just as in static optimization, we need some assumptions about differentiability and concavity:

SUFFICIENCY: *Suppose $f(t,x,u)$ and $g(t,x,u)$ are both differentiable functions of x, u in the problem*

$$\max_{u} \int_{t_0}^{t_1} f(t,x,u)dt$$

subject to

$$\dot{x} = g(t,x,u)\,,\quad x(t_0) = x_0\,.$$

Then, if f and g are jointly concave in x,u, and $\lambda \geq 0$ for all t, the necessary conditions are also sufficient.

PROOF. Suppose $x^*(t)$, $u^*(t)$, $\lambda(t)$ satisfy the necessary conditions, and let f^* and g^* denote the functions evaluated along $(t,x^*(t),\ u^*(t))$. Then, the proof requires us to show that

$$D = \int_{t_0}^{t_1}\left(f^* - f\right) \geq 0$$

for any feasible $f \neq f^*$. Since f is concave, we have

$$f^* - f \geq (x^* - x)f_x^* + (u^* - u)f_u^*\,,$$

and hence

$$D \geq \int_{t_0}^{t_1} \left[(x*-x)f_x^* + (u*-u)f_u^* \right] dt$$

$$= \int_{t_0}^{t_1} \left[(x*-x)(-\lambda g_x^* - \dot{\lambda}) + (u*-u)(-\lambda g_u^*) \right] dt$$

These substitutions come from the necessary conditions, C.1 and C.2.

$$= \int_{t_0}^{t_1} \lambda \left[-(x*-x)g_x^* - (x*-x)\frac{\dot{\lambda}}{\lambda} - (u*-u)g_u^* \right] dt .$$

Consider the second term,

$$\int_{t_0}^{t_1} -(x*-x)\dot{\lambda} dt ,$$

and integrate by parts to obtain

$$-\lambda(x*-x)\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \lambda(\dot{x}*-\dot{x}) dt .$$

Noting that $\lambda(t_1) = 0$ $[why?]$, and $x*(t_0) = x(t_0) = x_0$, the first term vanishes. Then, the equation of motion for the state variables, $\dot{x} = g$, gives us

$$\int_{t_0}^{t_1} \lambda(\dot{x}*-\dot{x}) dt = \int_{t_0}^{t_1} \lambda(g*-g) dt .$$

Hence,

$$D \geq \int_{t_0}^{t_1} \lambda \left[g*-g - (x*-x)g_x^* - (u*-u)g_u^* \right] dt .$$

But as g is concave, we have

$$g*-g > (x*-x)g_x^* + (u*-u)g_u^* ,$$

so the expression in square brackets is positive. then, noting that $\lambda \geq 0$ by assumption, we have $D \geq 0$, which completes the proof. $\bullet$

Note that $g(.)$ need not be concave. An alternative set of sufficiency conditions is that

- f is concave
- g is convex
- $\lambda(t) \leq 0$ for all t.

These are reversed

In this case, we would have $g*-g < (x*-x)g_x^* + (u*-u)g_u^*$ but, as $\lambda(t) \leq 0$, we still have $D \geq 0$.

For a minimization problem, we require $f(.)$ to be convex. This is because we can write

$$\min \int_{t_0}^{t_1} f(x,u,t)dt \text{ as } \max \int_{t_0}^{t_1} -f(x,u,t)dt ,$$

and if f is convex than $-f$ is concave, yielding the same sufficiency conditions as in the maximization problem.

Finally, note that these sufficiency conditions ensure that

- For a maximization problem, $\partial^2 H / \partial u^2 < 0$,

- For a minimization problem, $\partial^2 H / \partial u^2 > 0$,

at every point along the optimal path.

EXAMPLE 5.1 (*Making use of the sufficiency conditions*). Sometimes, the sufficiency conditions can be more useful than simply checking we have found a maximum. They can also be used directly to obtain the solution, as this example shows. The rate at which a new product can be sold is $f(p(t))g(Q(t))$, where p is price and Q is cumulative sales. Assume $f'(p) < 0$ and

$$g'(Q) \begin{cases} > 0 & \text{for } Q < Q_1 \\ < 0 & \text{for } Q > Q_1 \end{cases}.$$

That is, when relatively few people have bought the product, $g' > 0$ as word of mouth helps people learn about it. But when Q is large, relatively few people have not already purchased the good, and sales begin to decline. Assume further that production cost, c, per unit is constant. What is the shape of the path of price, $p(t)$ that maximizes profits over a time horizon T?

Formally, the problem is

$$\max_p \int_0^T (p - c)f(p)g(Q)dt \,,$$

subject to

$$\dot{Q}(t) = f(p)g(Q) \,. \tag{5.1}$$

Form the Hamiltonian:

$$H = [p - c]f(p)g(Q) + \lambda f(p)g(Q) \,.$$

The necessary conditions are

$$\frac{\partial H}{\partial p} = g(Q)\Big[f'(p)(p - c + \lambda) + f(p)\Big] = 0 \tag{5.2}$$

$$\frac{\partial H}{\partial Q} = g'(Q)f(p)[p - c + \lambda] = -\dot{\lambda} \tag{5.3}$$

plus the resource constraint, (5.1). We will use these conditions to characterize the solution qualitatively. From (5.2), we have

$$\lambda = -\frac{f(p)}{f'(p)} - p + c \,. \tag{5.4}$$

Again, our solution strategy is to get rid of the costate variable. Differentiate (5.4) with respect to time,

$$\dot{\lambda} = -\dot{p}\left[2 - \frac{f(p)f''(p)}{\left[f'(p)\right]^2}\right]. \tag{5.5}$$

Now substitute (5.4) into (5.3) to obtain another expression involving $\dot{\lambda}$ but not λ:

$$\dot{\lambda} = \frac{g^{/}(Q)[f(p)]^{2}}{f^{/}(p)} \, , \tag{5.6}$$

and equate (5.5) and (5.6) to remove $\dot{\lambda}$:

$$-\dot{p}(t)\left[2 - \frac{f(p)f^{//}(p)}{\left[f^{/}(p)\right]^{2}}\right] = \frac{g^{/}(Q)[f(p)]^{2}}{f^{/}(p)} \, . \tag{5.7}$$

We still don't have enough information to say much about (5.7). It is at this point that we can usefully exploit the sufficiency conditions. As we are maximizing H, we have the condition

$$\frac{\partial^{2}H}{\partial p^{2}} = g(Q)\left[f^{//}(p)(p - c + \lambda) + 2f^{/}(p)\right] < 0 \, . \tag{5.8}$$

Substitute (5.4) into (5.8) to remove λ:

$$g(Q)f^{/}(p)\left[2 - \frac{f(p)f^{//}(p)}{\left[f^{/}(p)\right]^{2}}\right] < 0 \, , \tag{5.9}$$

which must be true for *every* value of p along the optimal path. As $g(Q) > 0$ and $f^{/}(p) < 0$, the expression in square brackets in (5.9) must be positive. Thus, in (5.7) we must have

$$sign\left\{-\dot{p}(t)\right\} = sign\left\{\frac{g^{/}(Q)[f(p)]^{2}}{f^{/}(p)}\right\} \, ,$$

or

$$sign\left\{\dot{p}(t)\right\} = sign\left\{g^{/}(Q)\right\} \, .$$

That is, the optimal strategy is to keep raising the price until Q_{1} units are sold, and then let price decline thereafter.   ●

6. Infinite Planning Horizons

So far we have assumed that T is finite. But in many applications, there is no fixed end time, and letting $T \to \infty$ is a more appropriate statement of the problem. But then, the objective function

$$\max \int_0^\infty f(x, u, t)dt$$

may not exist because it may take the value $+\infty$. In such problems with infinite horizons, we need to discount future benefits:

$$\max \int_0^\infty e^{-\rho t} f(x, u, t)dt \, .$$

If ρ is sufficiently large, then this integral will remain finite even though the horizon is infinite.

Current and Present Value Multipliers

Given the following problem:

$$\max_u \int_0^\infty e^{-\rho t} f(x, u, t)$$

subject to $\dot{x}(t) = g(x, u, t) \, ,$

we can write the Hamiltonian in the usual way:

$$H = e^{-\rho t} f(x, u, t) + \lambda(t) g(x, u, t) \, ,$$

giving the following necessary conditions

$$e^{-\rho t} f_u + \lambda g_u = 0 \, , \tag{6.1}$$

$$e^{-\rho t} f_x + \lambda g_x = -\dot{\lambda} \, , \tag{6.2}$$

$$g(x, u, t) = \dot{x} \, .$$

Recall that $\lambda(t)$ is the shadow price of $x(t)$. Because the benefits are discounted in this problem, however, $\lambda(t)$ must measure the shadow price of $x(t)$ discounted to time 0.

That is, $\lambda(t)$ is the discounted shadow price or, more commonly, **the present value multiplier**. The Hamiltonian formed above is known as the **present value Hamiltonian**.

This is not the only way to present the problem. Define a new costate variable,

$$\mu(t) = e^{\rho t} \lambda(t) \,. \tag{6.3}$$

The variable $\mu(t)$ is clearly the shadow price of $x(t)$ evaluated at time t, and is thus known as the **current value multiplier**. Noting that

$$\dot{\lambda}(t) = \frac{d}{dt}\left[e^{-\rho t}\mu(t)\right] = e^{-\rho t}\dot{\mu}(t) - \rho e^{-\rho t}\mu(t) \,, \tag{6.4}$$

we can multiply (6.1) and (6.2) throughout by $e^{\rho t}$, and then substitute for $\lambda(t)$ using (6.3) and (6.4) to get

$$f_u + \mu(t)g_u = 0 \,, \tag{6.5}$$

$$f_x + \mu(t)g_x = -\dot{\mu}(t) + \rho\mu(t) \,, \tag{6.6}$$

$$g(x, u, t) = \dot{x} \,. \tag{6.7}$$

These are the necessary conditions for the problem when stated in terms of the current value multiplier. The version of the Hamiltonian used to generate these conditions is the current value Hamiltonian,

$$H = f(u, x, t) + \mu(t)g(u, x, t) \,,$$

The discounting term is simply omitted.

with necessary conditions:

$$\frac{\partial H}{\partial u} = 0 \,, \quad \frac{\partial H}{\partial x} = -\dot{\mu} + \rho\mu \,, \quad \text{and } \frac{\partial H}{\partial \mu} = \dot{x} \,.$$

Obviously, when $\rho = 0$, the current and present value problems are identical. For any $\rho \neq 0$, the two versions of the Hamiltonian will always generate identical time paths for u and x. The time paths for λ and μ are different, however, because they are measuring different things. However, it is easy to see that they are related by

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = \frac{\dot{\mu}(t)}{\mu(t)} - \rho \,.$$

Economists arbitrarily choose between these two methods.

The Transversality Conditions

There has been much debate in the past about whether the various transversality conditions used for finite planning horizon problems extend to the infinite-horizon case. The debate arose as a result of some apparent counterexamples. However, it is now known that these counterexamples could be transformed to generate equivalent mathematical problems in which the endpoint conditions were different from what had been believed. I mention this only because some texts continue to raise doubts about the validity of the transversality conditions in infinite horizon problems.

TABLE 6.1 *Transversality Conditions*		
FINITE HORIZON CONDITION	INFINITE HORIZON CONDITION	
	PRESENT VALUE	CURRENT VALUE
$\lambda(T){=}0$	$\lim\limits_{t\to\infty} \lambda(t) = 0$	$\lim\limits_{t\to\infty} e^{-\rho t}\mu(t) = 0$
$\lambda(T)(y(T) - y_{\min}) = 0$	$\lim\limits_{t\to\infty} \lambda(t)(y(t) - y_{\min}) = 0$	$\lim\limits_{t\to\infty} e^{-\rho t}\mu(t)(y(t) - y_{\min}) = 0$
$\lambda(T)y(T){=}0$	$\lim\limits_{t\to\infty} \lambda(t)y(t) = 0$	$\lim\limits_{t\to\infty} e^{-\rho t}\mu(t)(y(t) - y_{\min}) = 0$
$H(T){=}0$	$H(T){=}0$	$e^{-\rho t}H(T){=}0$
$(T{-}T_{max})H(T){=}0$	$T_{\max} \to \infty$, so there is no equivalent problem.	

In fact, the only complications in moving to the infinite-horizon case are (i) Instead of stating a condition at T, we must take the limit as $t \to \infty$. (ii) The transversality condition is stated differently for present value and current value Hamiltonians. Because no further difficulties are raised, we simply provide the equivalent transversality conditions in Table 6.1.

EXERCISE 6.1. *Find the path of $x(t)$ that maximizes*

$$V = \int_{t_0}^{\infty} e^{-\rho t} \ln x(t) dt \ ,$$

subject to $\dot{m}(t) = \beta m(t) - x(t)$, $m(t_0) = m_0 > 0$, $\lim_{t \to \infty} m(t) = 0$. *Assume that* $\rho > \beta$.

EXERCISE 6.2. *There is a resource stock, $x(t)$ with $x(0)$ known. Let $p(t)$ be the price of the resource, let $c(x(t))$ be the cost of extraction when the remaining stock is $x(t)$, and let $q(t)$ be the extraction rate. A firm maximizes the present value of profits over an infinite horizon, with an interest rate of r.*

(a) *Assume that extraction costs are constant at c. Show that, for any given $p(0) > c$, the price of the resource rises over time at the rate $\alpha(t)r$ for some function $\alpha(t)$. Show that $\alpha(t) < 1$ for all t.*

(b) *Assume $c'(x(t)) < 0$. Derive an expression for the evolution of the price (it may include an integral term) and compare your answer with part (a).*

(c) *How do we produce a time path for price without considering demand? What does your solution tell you about the rate of extraction?*

(d) *Over the last 20 years, there has been no tendency for the price of oil to rise, despite continued rapid rates of extraction. How does your answer to part (b) shed light on the possible explanations for the failure of the price of oil to rise?*

7. Infinite Horizon Problems and Steady States

In many economic applications, not only is the planning horizon infinitely long, time does not enter the problem at all except through the discount rate. That is, the problem takes the form

$$\max \int_0^\infty e^{-\rho t} f(x, u) dt \text{, subject to } \dot{x} = g(x, u). \tag{7.1}$$

Such problems are know as **infinite-horizon autonomous problems**. There is an immediate analogy to the autonomous equations and systems of equations studied in the differential equations module. In these problems, the transversality conditions are often replaced by the assumption that the optimal solution approaches a steady state in which $\dot{x}^* = g(x^*, u^*) = 0$, and u^* is constant. Analysis of these models must always be concerned with the existence and stability of steady states. The techniques used to analyze them are exactly the same as the techniques used to analyze autonomous differential equations.

When Do Steady States Exist?

Using the current value Hamiltonian, the necessary conditions for (7.1) are

$$H_u = f_u + \mu g_u = 0, \tag{7.2}$$

$$H_x = f_x + \mu g_x = -\dot{\mu} + \rho\mu, \tag{7.3}$$

$$\dot{x} = g(x, u). \tag{7.4}$$

Assume that $H_{uu} \leq 0$, to ensure a maximum. A steady state is characterized by a set of values, (x^*, u^*, μ^*) such that $\dot{x} = \dot{u} = \dot{\mu} = 0$ (Note that the current value multiplier is constant but the present value multiplier is not; hence it is useful to think about steady-states in terms of the current value Hamiltonian). To define the steady state, we therefore set $\dot{x} = \dot{\mu} = 0$ in (7.3) and (7.4)

$$f_x + \mu g_x = \rho\mu \tag{7.5}$$

$$g(x, u) = 0, \tag{7.6}$$

while noting that (7.2) must also be satisfied. If we can find triplets (x^*, u^*, μ^*) that satisfy (7.2), (7.5), and (7.6), then at least one steady state exists. If we can show that only one solution exists, then the steady state is unique. If there are no solutions, there are no steady states.

In many journal articles, you will find that, when a problem is set up, a number of assumptions about the arbitrary functions are given. It then just happens to turn out that the problem leads to a nice, unique steady state. In practice, what the authors have done is study equations (7.2), (7.5) and (7.6) and asked what assumptions are necessary to ensure a unique solution to the equations. These assumptions are then presented and justified at the beginning of the paper rather than at the end. It is best to see these ideas in practice.

EXAMPLE 7.1. (Making sure a steady-state exists) Let a renewable resource stock evolve according to $\dot{x}(t) = f(x(t)) - q(t)$, where $f(x(t))$ is a growth function and $q(t)$ is the extraction rate. Given extraction costs $c(x(t))$ and a constant interest rate of r, solve the profit maximization problem for a representative competitive firm. Provide sufficient conditions for the existence of a unique steady state.

The profit-maximizing problem is

$$\max_{q} \int_{0}^{\infty} \left(p(t) - c(x(t)) \right) q(t) e^{-rt} dt \;,$$

subject to

$$\dot{x}(t) = f(x(t)) - q(t)\,. \tag{7.7}$$

The first-order conditions (using the current value Hamiltonian) are (7.7) plus

$$p(t) - c(x(t)) = \mu(t)\,, \tag{7.8}$$

and

$$-c'\left(x(t) \right) q(t) + \mu(t) f'\left(x(t) \right) = -\dot{\mu}(t) + r\mu(t)\,. \tag{7.9}$$

A steady state is characterized by constant values for the shadow price, the control and the state variable. From (7.8), this implies that a steady state can only exist if

A.1: p is constant,

so this will constitute one of our key assumptions. Setting $\dot{\mu} = 0$ in (7.9), and using the facts that in the steady state we must have $p - c(x^*) = \mu^*$ (from (7.8)) and $q^* = f(x^*)$ (from (7.7)), we obtain

$$c^{/}(x^*)f(x^*) = \left(p - c(x^*)\right)\left(f^{/}(x^*) - r\right) \tag{7.10}$$

Thus, we need to consider conditions that ensure (4) has a unique solution for x^*. We begin first with economically plausible assumptions about $c(x)$ and $f(x)$, and see if they provide us with enough information.

A.2: $c(x)$ is the extraction cost, and this can reasonably be expected to be lower the larger the stock. i.e. $c^{/}(x) < 0$.

A.3: $f(x) > 0$, but $f^{/}(x) < 0$. The resource recovery rate is positive, but declining in the size of the stock. This assumption may not be true if the resource stock becomes so low as to face extinction (e.g. in fisheries).

A.4: $p > c(x^*)$ so that the price is high enough to justify extraction in the steady state. If this assumption is not true, then the steady-state will involve zero activity.

These assumptions ensure that both the left-hand side and right-hand side of (7.10) are negative. Differentiating the left-hand side, we have

$$\frac{d}{dx^*}\left[c^{/}(x^*)f(x^*)\right] = f(x^*)c^{//}(x^*) + c^{/}(x^*)f^{/}(x^*),$$

which is unambiguously positive if we further assume that

A.5: $c^{//}(x) > 0$ (i.e. extraction costs decline with the size of the stock at a declining rate). This seems like a reasonable assumption, and it implies that the left hand side of (7.10) is increasing in x^*.

Similarly, if we assume that $f^{//}(x) > 0$, the right hand side of (7.10) can be seen to be decreasing in x^*. Thus, there will be exactly one value of x^* that satisfies (7.10). But is this a feasible (i.e. positive) value of x^*? Because the left hand side is strictly negative and increasing, if there is a positive value of x such that the right hand side of (7.10) is zero, then we will have a unique positive steady state stock. But this requires only that costs rise high enough to reach p at some positive but small stock level. Hence, the final assumption is that

A.6: $\lim_{x \to 0} c(x) > p$.

Finally, it is easy to verify that $H_{qq} \leq 0$ for all q, so that the solution is indeed a maximum.

Hence, to establish uniqueness, we have made six assumptions. Five of these seem reasonable, but assumption A.3. may not always be reasonable. Of course, it is easy to intuitively work out that if A.3 is violated, then one could have a second steady state with $x^*{=}0$ (i.e. extinction).   •

Stability of Steady States

Usually, we will be very interested in how one approaches a steady-state. In some cases, steady states may be unstable – one may not be able to reach them at all unless the initial conditions drop you right on them. These unstable steady states are usually not of economic interest. If there are multiple steady states, it is usually possible to exclude one or more of them by rejecting those that are unstable.

We study the stability of steady states using the graphical and algebraic techniques that we studied in the differential equations module. We will not repeat that material here.

EXERCISE 7.1. *Continuing the renewable resource stock problem of Example 7.1, assume now that $f(x(t)) = x(t)(1 - x(t))$, and that the extraction rate is given by*

$$q(t) = 2x(t)^{1/2} n(t)^{1/2},$$

where $n(t)$ is labor effort with a constant cost of w per unit. Assume further that this resource is fish that are sold at a fixed price p. The optimization problem is to choose $n(t)$ to maximize discounted profits.

(a) *Derive the necessary conditions and interpret them verbally.*

(b) *Draw the phase diagram for this problem, with the fish stock and the shadow price of fish on the axes.*

(c) *Show that, if p/w is large enough, the fish stock will be driven to zero, while if p/w is low enough there is a steady state with a positive stock.*

8. Applications from Growth

Optimal control is a very common tool for the analysis of economic growth. This section provides examples and exercises in this sphere.

EXAMPLE 8.1 (*The Ramsey-Cass-Koopmans model*). Suppose family utility is given by

$$\int_0^\infty e^{-\rho t} U(c(t)) dt$$

where U is an increasing strictly concave function. The household budget constraint is

$$\dot{k}(t) = r(t)k(t) + w(t) - c(t) \,.$$

Households choose $c(t)$ to maximize discounted lifetime utility. Here we derive the steady state for competitive economy consisting of a large number of identical families.

The current value Hamiltonian is

$$H = U(c(t)) + \lambda(t)\big(r(t)k(t) + w(t) - c(t)\big) \,,$$

which gives the following first-order conditions:

$$U'(c(t)) - \lambda(t) = 0 \,, \tag{8.1}$$

$$\lambda(t)r(t) = -\dot{\lambda}(t) + \rho\lambda(t) \,, \tag{8.2}$$

plus the resource constraint.

Differentiating (8.1),

$$\dot{\lambda}(t) = U''(c(t))\dot{c}(t) \,,$$

and substituting into (8.2) yields

$$\frac{\dot{c}(t)}{c(t)} = \left[-\frac{1}{c(t)} \frac{U'(c(t))}{U''(c(t))} \right] (r(t) - \rho) \,.$$

The term in square brackets can be interpreted as the inverse of the elasticity of marginal utility with respect to consumption, and is a positive number. Thus, if the interest rate exceeds the discount rate, consumption will grow over time. The rate of growth is larger the smaller is the elasticity of marginal utility. The reason for this is as

follows. Imagine that there is a sudden and permanent rise in the interest rate. It is possible then to increase overall consumption by cutting back today to take advantage of the higher return to saving. Consumption will then grow over time, eventually surpassing its original level. However, if the elasticity of marginal utility is high, a drop in consumption today induces a large decline in utility relative to more modest gains that will be obtained in the future (note that the elasticity is also a measure of the curvature of the utility function). Thus, when the elasticity is high, only a modest change in consumption is called for.

So far, we have solved only for the family's problem. To go from there to the competitive equilibrium, note that the interest rate will equal the marginal product of capital, while the wage rate will equal the marginal product of labor. The latter does not figure into the optimization problem, however. Let the production function be given by $y(t) = f(k(t))$. Setting $r(t) = f'(k(t))$, we have

$$\frac{\dot{c}(t)}{c(t)} = \left[-\frac{1}{c(t)} \frac{U'(c(t))}{U''(c(t))} \right] \left(f'k(t) - \rho \right). \tag{8.3}$$

In the steady state, consumption and capital are constant. Setting consumption growth to zero in (8.3) implies $f'(k^*) = \rho$. Now, there is no depreciation of capital in this economy, so if capital is constant, all output is being consumed. Thus, we also have $c^* = f(k^*)$.

Figure 8.1 provides the phase diagram for this model. The system is saddle-path stable. You will recall from Chapter 1 that in saddle-path stable systems only initial choices that place the system on the single stable path will lead to the unique steady state. It was claimed then that in optimization problems the transversality condition ensures that this path is selected out of all possible solutions to pair of differential equations. The appropriate transversality condition for the Ramsey model is one that imposes a non-negativity constraint on capital (i.e. a no-Ponzi scheme). That is,

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t) k(t).$$

If the terminal capital has a positive present value (from the perspective of time 0), then you must consume it all, so $k(t)$ is zero. Otherwise, it must have zero present value, so you are indifferent about leaving some on the table unexploited.

Any path to the left of the saddle path in Figure 8.1 leads us off into negative capital, and this violates the no-Ponzi scheme condition. Hence, none of these paths can be feasible (but they would be optimal if there were no non-negativity constraint. Any path to the right of the saddle path leads us off into negative consumption (or down to zero consumption, when we impose the logical non-negativity constraint on consumption). Along such paths, the capital stock grows forever. This cannot be optimal because the capital is *never* consumed: one could always convert some of this excess capital into concumption and raise utility. Consider the path **aAa**, and assume that at time T the point **A** is reached. Here the marginal utility of consumption approaches infinity. One can always take a small fraction ε of the capital at time T, and consume it, with present value $e^{-\rho T} MU(c) \big|_{c=0} \varepsilon k(T) = \infty$. Because this capital is never consumed later, there is no consequence for future consumption, so lifetime utility is unambiguously increased by consuming some of the capital. Hence, any path to the right of the saddle path cannot be optimal.   •
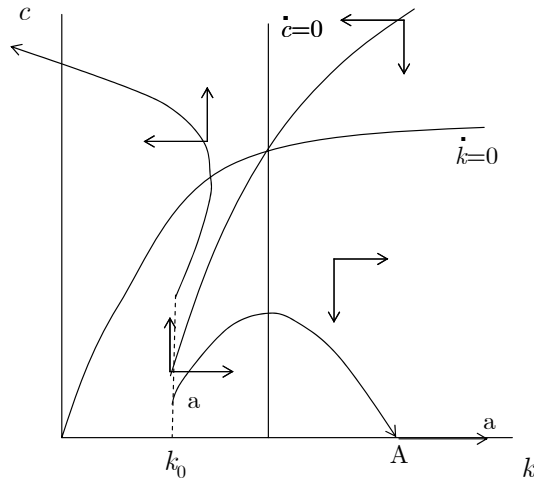

FIGURE 8.1

EXERCISE 8.1 (*Fish as a type of capital*). *Any stock variable can be considered a type of capital and analyzed in the manner of Ramsey, Cass and Koopmans. The number of fish in a certain lake evolves according to*

$$\dot{n}(t) = an(t) - b(n(t))^2 - c(t),$$

where a>0, and b>0. That is, the growth arte of the fish population is the net reproduction rate less the number of fish consumed. The net reproduction rate is the birth rate minus the death rate and is a concave function of the number of fish in the lake. Assume society maximizes the discounted utility of consumption,

$$\int_0^\infty \ln(c(t))e^{-\rho t}dt \ .$$

(a) *Derive the first-order conditions and explain their meaning.*

(b) *Derive the steady state conditions and interpret verbally.*

(c) *Draw a phase diagram for this problem with c and n on the axes. Show the transition toward the steady state from an initial condition $n_0 = a/b$. Describe what is happening to the price of fish and the fish population.*

EXERCISE 8.2 (The Ramsey-Cass-Koopmans model with a pollution externality). *Suppose family utility is given by*

$$\int_0^\infty \left(U(c(t)) - V(y(t))\right)e^{-\rho t}dt \tag{8.4}$$

where U is an increasing strictly concave function and V is an increasing strictly convex function. $-V(y(t))$ is the disutility of pollution associated with production.

(a) Decentralized economy. *The household budget constraint is*

$$\dot{k}(t) = r(t)k(t) + w(t) - c(t).$$

Households choose c(t) to maximize (8.4), and ignore the effects of k(t) on y(t). Derive the steady state of the competitive economy.

(b) Command economy. *The social planner's resource constraint is*

$$\dot{k}(t) = f(k(t)) - c(t) \, ,$$

and the planner maximizes (8.4) subject to the constraint, taking into account the fact that $y(t) = f(k(t))$. Derive the steady state conditions and compare your answer with part (a).

EXERCISE 8.3 (The Ramsey-Cass-Koopmans model with leisure). *Consider a competitive economy populated by identical infinitely-lived individuals whose utility functions are given by*

$$\int_0^\infty \big(U(c(t)) + V(T - h(t))\big) e^{-\rho t} dt \, ,$$

where U and V are increasing strictly concave functions. $c(t)$ is consumption, T is the number of hours in the day, $h(t)$ is the number of hours spent working, so that T-h(t) is the number of hours of leisure per day. The marginal utilities of consumption and leisure are positive and diminishing. An individual's budget constraint is given by

$$\dot{k}(t) = r(t)k(t) + w(t)h(t) + z(t) - c(t) - \tau w(t)h(t) \, ,$$

where $k(t)$ is capital holdings, $r(t)$ is the rental rate on capital, $w(t)$ is the hourly wage, $z(t)$ is a transfer payment from the government, and τ is the tax rate on labor income.

(a) *Derive the optimality conditions for the individual.*

The per capita net production function for this economy is $y(t) = f(k(t), h(t))$, where f is strictly concave in each input, homogenous of degree one, and the derivatives satisfy $f_{hk} > 0$ and $f_{hh} f_{kk} - (f_{hk})^2 > 0$.

(b) *Assume that markets are competitive and that the government maintains a balanced budget. Find and interpret the three conditions that determine steady-state per capita consumption, capital and hours as a function of τ.*

(c) *Find expressions for the effect of an increase in τ on the steady-state values of c, k, and h. Interpret these results.*

EXAMPLE 8.2 (*The Lucas-Uzawa model*). The Solow framework and the Ramsey-Cass-Koopmans framework cannot be used to account for the diversity of income levels and rates of growth that have been observed in cross-country data. In particular, if parameters such as the rate of population growth vary across countries, then so will income levels. However, the modest variations in parameter values that are observed in the data cannot generate the large variations in observed income. Moreover, in a model with just physical capital, a curious anomaly is apparent. Countries have low incomes in the standard model because they have low capital-labor ratios. However, because of the shape of the neoclassical production function, a low capital-labor ratio implies a high marginal productivity of capital. But if the returns to investment are high in poor countries, why is it that so little investment flows from rich to poor countries?

Lucas (1988) attempts to tackle these anomalies by developing a model of endogenous growth driven by the accumulation of human capital. In this example, we describe the model and derive the balanced growth equilibrium. Much of Lucas' paper (and it is in my opinion one of the most beautifully written papers of the last twenty years) concerns the interpretation of the model – the reader should refer to Lucas' paper for those details.

Utility is given by

$$\max_{c} \int_0^\infty e^{-\rho t} \left(\frac{c(t)^{1-\sigma} - 1}{1-\sigma} \right) N(t) dt \ ,$$

where $N(t)$ is population, assumed to grow at the constant rate λ. (As an exercise, you should verify that the instantaneous utility function takes the form $\ln(c(t))$ when $\sigma \to 1$).

The resource constraint is

$$\dot{K}(t) = h_a(t)^\gamma K(t)^\beta \left(u(t) h(t) N(t) \right)^{1-\beta} - N(t) c(t) \ , \tag{8.5}$$

and the evolution of human capital satisfies

$$\dot{h}(t) = \delta(1 - u(t)) h(t) \ , \tag{8.6}$$

where $1 - u \in (0,1)$ is the fraction of an individual's time spent accumulating human capital, and u is the fraction of time spent working; $h(t)$ is the individual's current level of human capital, and $h_a(t)$ is the economy-wide average level of human capital; $K(t)$ is

aggregate physical capital. There is an externality in this model: if an individual raises his human capital level, he raises the average level of human capital and hence the productivity of everyone else in the economy.

There is a trick to dealing with an externality when there is but a representative agent (and so $h(t)=h_a(t)$). The social planner takes the externality into account, while the individual does not. So, in the planner's problem, we first set $h_a(t)= h(t)$ and then solve the model. In the decentralized equilibrium, we solve the model taking $h_a(t)$ as given and then after obtaining the first-order conditions we set $h_a(t)= h(t)$. In this example, we will just look at the planner's problem.

The current value Hamiltonian is

$$H(t) = \frac{N(t)}{1-\sigma}\left(c(t)^{1-\sigma} - 1\right) + \theta_1(t)\left(K(t)^\beta\left(u(t)N(t)h(t)\right)^{1-\beta} h(t)^\gamma - N(t)c(t)\right)$$

$$+\theta_2(t)\left(\delta(1 - u(t))h(t)\right).$$

There are two controls, $u(t)$ and $c(t)$, two state variables, $h(t)$ and $k(t)$, and two costate variables in this problem. The first-order conditions (dropping time arguments for ease of exposition) are

$$\frac{\partial H}{\partial c} = c^{-\sigma} - \theta_1 = 0 \ , \tag{8.7}$$

$$\frac{\partial H}{\partial u} = \theta_1 f_u - \theta_2\delta h = 0 \ , \tag{8.8}$$

$$\frac{\partial H}{\partial k} = \theta_1 f_k = -\dot{\theta}_1 + \rho\theta_1 \ , \tag{8.9}$$

$$\frac{\partial H}{\partial h} = \theta_1 f_h + \theta_2\delta(1 - u) = -\dot{\theta}_2 + \rho\theta_2 \ , \tag{8.10}$$

plus the resource constraints. The derivatives of f in these expression refer to the production function $y = f(K,u,N,h) = K^\beta(uNh)^{1-\beta}h^\gamma$. Let us interpret these first-order conditions:

- $\theta_1 = c^{-\sigma}$. The shadow price of capital equals the marginal utility of consumption. Why? Because the marginal unit of capital can be converted to consump-

tion costlessly at a one for one rate, the two uses for the marginal unit must have equal value.

- $\theta_1 f_u = \theta_2 \delta h$. The marginal value product of effort in labor equals its marginal value product in the accumulation of human capital.

- $f_k + \dot{\theta}_1 / \theta_1 = \rho$. The net marginal product of capital, plus any change in the marginal value of capital, equals the discount rate. Think of this as dividends plus capital gains must equal the rate of return on a risk free asset.

- $(\theta_1 / \theta_2) f_h + \delta(1-u) + \dot{\theta}_2 / \theta_2 = \rho$. The returns to human capital plus any change in the marginal value of human capital equals the discount rate. Note that the term θ_1 / θ_2 accomplishes a change of units of measurement: it expresses the productivity of human capital in terms of consumption utility.

Lucas analyzes a version of the steady state known as the *balanced growth path*. This is where all variables grow at a constant, although not necessarily equal, rate. The basic line of Lucas' analysis is as follows:

First, note that u has an upper and lower bound, so that in the balanced growth path it must be constant: $\dot{u} = 0$. Next, let $k = K/N$. Then, from (8.5),

$$\dot{k} = h^{1-\beta+\gamma} k^\beta u (1-\beta) - c - \lambda k$$

$$= y - c - \lambda k .$$

Because $\dot{k}$ is written as the sum of y, c, and λk, *each* of these variable must be growing at the same rate for the growth rate of capital per capita to be constant. Now, take the production function and divide through by n. Then, differentiating with respect to time, we have

$$\frac{\dot{y}}{y} = \beta \frac{\dot{k}}{k} + (1-\beta+\gamma)\frac{\dot{h}}{h} + (1-\beta)\frac{\dot{u}}{u} .$$

$\underset{=0}{}$

Hence,

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} = \frac{\dot{c}}{c} \quad \text{and} \quad \frac{\dot{y}}{y} = (1-\beta)\frac{\dot{k}}{k} - (1-\beta+\gamma)\frac{\dot{h}}{h} .$$

Denote $\dot{y}/y = E$ and $\dot{h}/h = V$. Then, it is easy to see that

$$E = \beta E + (1 - \beta + \gamma)V \quad \Rightarrow \quad E = \frac{1 - \beta + \gamma}{1 - \beta}V > V,$$

which shows that output, consumption and physical capital per capita all grow at a common rate that is faster than human capital. We now need to solve for V, which depends on preferences. From (8.8) we have

$$\ln \theta_2 + \ln \delta + \ln h = \ln(\theta_1) + \ln f_u$$

$$= \ln \theta_1 + \ln(1 - \beta) + \beta \ln K - \beta \ln u + (1 - \beta) \ln N + (1 - \beta + \gamma) \ln H.$$

Differentiating, we get

$$\frac{\dot{\theta_1}}{\theta_1} + \beta \frac{\dot{K}}{K} + (1 - \beta)\lambda + (1 - \beta + \gamma)V = \frac{\dot{\theta_2}}{\theta_2} + V. \tag{8.11}$$

Now, note that $\dot{K}/K = E + \lambda$, $\dot{\theta_1}/\theta_1 = -\sigma \dot{c}/c = -\sigma E$, and $E = (1 - \beta + \gamma)V/(1 - \beta)$. Using all these in (8.11), we get

$$\frac{\dot{\theta_2}}{\theta_2} = -\sigma \left(\frac{1 - \beta + \gamma}{1 - \beta} \right) V + \frac{\gamma}{1 - \beta}V + \lambda. \tag{8.12}$$

We now need to remove $\dot{\theta_2}/\theta_2$ from (8.12). Using (8.10), we get

$$\frac{\dot{\theta_2}}{\theta_2} = \rho - \frac{\theta_1}{\theta_2 f_h} - \delta(1 - u)$$

$$= \rho - \frac{\delta h f_h}{f_u} - V \qquad \text{(from (8.8))}$$

$$= \rho - \frac{\delta(1 - \beta + \gamma)}{1 - \beta} - V$$

$$= \rho - \frac{(1 - \beta + \gamma)(\delta - V)}{1 - \beta} - V. \tag{8.13}$$

where use was made in the last line of the fact that $\delta u = \delta - V$. Finally, combining (8.12) and (8.13) yields

$$V = \frac{1}{\sigma} \left[\delta + \frac{(1 - \beta)}{(1 - \beta + \gamma)}(\lambda - \rho) \right].$$

The growth rate of human capital is increasing in the marginal product of effort in human capital accumulation δ, increasing in the population growth rate, λ, and decreasing in the discount rate, ρ. The Lucas model, by endogenizing the accumulation of capital, also provides reasons why small differences in parameters can lead to long-run differences in growth rate and, ultimately, large differences in per capita income.

> EXERCISE 8.4 (The decentralized equilibrium in Lucas' model). *Repeat the previous analysis for the decentralized equilibrium and compare the decentralized balanced growth path with the planner's problem. How big a subsidy to human capital accumulation (*i.e. schooling*) is necessary to attain the social optimum? On what does the optimal subsidy depend?*

EXAMPLE 8.3 (*Learning by Doing*). The idea that firms become more productive simply because past experience in production offers a learning experience has a long empirical history, but was first formalized by Arrow (1962). There is certainly considerable ev9idence that technology in many firms and industries exhibits a learning curve, in which productivity is related to past production levels. While it is less self evident, at least to me, that this should be attributed to passive learning that results simply fro familiarity with the task at hand, the concept of growth generated by learning by doing has received widespread attention. This example present some simple mechanics of an aggregate learning model.

Preferences of the representative consumer are assumed to satisfy the isoelastic form

$$U = \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \,,$$

while the aggregate resource constraint is

$$\dot{K}(t) = K(t)^\beta H(t)^\alpha L^{1-\beta} - C(t) \,. \tag{8.14}$$

where output is $y(t) = K(t)^\beta H(t)^\alpha L^{1-\beta}$. $H(t)$ continues to represent human capital, but it may be less confusing to use the term 'knowledge capital'. Instead of requiring that work-

ers withdraw from manufacturing in order to increase $H(t)$, learning by doing relates human capital to an index of past production:

$$H(t) = \frac{dz(t)}{L}, \quad \text{where} \ z(t) = \int_0^t y(s)ds \ \text{and} \ H(0)=1.$$

In order to get this optimization problem into a standard form, it is useful to obtain an expression for $H(t)$ that does not involve an integral over y. By Liebnitz' rule,

$$\dot{H}(t) = \frac{\delta y(t)}{L(t)} = \delta K(t)^\beta H(t)^\alpha L^{-\beta} \quad .$$

Note that there is less choice in this model than in the Lucas-Uzawa model. Learning is a passive affair and is not a direct choice variable. This feature of the model simplifies the problem because we can solve this differential equation and then substitute for $H(t)$ into (8.6) to get a Hamiltonian involving only one state variable. Although the differential equation is nonlinear, it can easily be solved using a change of variables $x(t)=H(t)^{1-\alpha}$ (left as an exercise), yielding the solution

$$H(t) = \left[1 + (1-\alpha)L^{-\beta}\delta \int_0^t K(s)^\beta \, ds\right]^{\frac{1}{1-\alpha}}.$$

(You can verify this by direct differentiation). Now, substituting the solution into (8.14) and combining with the preference structure allows us to form the current-value Hamiltonian:

$$J(t) = \frac{c(t)^{1-\sigma}}{1-\sigma} + \lambda(t)\left[L^{1-\beta}K(t)^\beta\left(1 + (1-\alpha)L^{-\beta}\delta \int_0^t K(s)^\beta \, ds\right)^{\frac{\alpha}{1-\alpha}} - c(t)\right]. \tag{8.15}$$

In contrast to the model of human capital, knowledge may well be appropriable by firms. If this is the case, firms would enjoy dynamic increasing returns to scale, and a competitive equilibrium could not be maintained. This leaves us two options. One is to model explicitly the imperfect competition that would ensure form dynamic scale economies. The second is to state explicitly that the accumulation of knowledge is entirely external to firms. We will do the second here, and doing so implies that in the maximization of

(8.15) the integral term (which reflects the external knowledge capital) is taken as given by the representative agent.

The first-order conditions are

$$c(t)^{-\sigma} = \lambda(t) \tag{8.16}$$

$$\rho - \beta L^{1-\beta} K(t)^{\beta-1} \left[1 + (1-\alpha)L^{-\beta}\delta \int_0^t K(s)^\beta \, ds \right]^{\frac{\alpha}{1-\alpha}} = \frac{\dot\lambda(t)}{\lambda(t)}, \tag{8.17}$$

plus the resource constraint. We follow the usual procedure for analyzing these equations. Differentiate (8.16) with respect to time and substitute into (8.17):

$$\sigma \frac{\dot c(t)}{c(t)} = \beta L^{1-\beta} K(t)^{\beta-1} \left[1 + (1-\alpha)L^{-\beta}\delta \int_0^t K(s)^\beta \, ds \right]^{\frac{\alpha}{1-\alpha}} - \rho \tag{8.18}$$

As in the Lucas-Uzawa model, a balanced growth path is characterized by a constant rate of growth of consumption and physical capital. That is, the left-hand side of (8.18) is constant along the balanced growth path. Thus, differentiate the right hand side with respect to time and set to zero. Some rearrangement gives the following condition for the growth rate of physical capital:

$$(1-\beta)\frac{\dot K(t)}{K(t)} = \frac{\alpha\delta L^{-\beta} K(t)^\beta}{1 + (1-\alpha)L^{-\beta}\delta \int_0^t K(s)^\beta \, ds}. \tag{8.19}$$

But the balanced growth is also characterized by a constant growth rate of physical capital. Now, the only way for the left hand side of (8.19) to be constant is if capital grows at just the rate that means the two terms involving K on the right hand side growth at exactly the same rate. This requires that

$$\beta \frac{\dot K(t)}{K(t)} = \frac{(1-\alpha)\delta L^{-\beta} K(t)^\beta}{1 + (1-\alpha)L^{-\beta}\delta \int_0^t K(s)^\beta \, ds}. \tag{8.20}$$

Combining (8.19) and (8.20), we find that these equations can only be satisfied if $\alpha=1-\beta$. Hence, for a balanced growth path to exit, the parameters of the model have to be re-

stricted. If $\alpha > 1 - \beta$, the growth rate increases without bound; if $\alpha < 1 - \beta$, the growth rate declines to zero.

So let us assume that $\alpha = 1 - \beta$ and solve for the long-run rate of growth. This is still not easy. Using $\alpha = 1 - \beta$ in (8.18), we get

$$\left(\frac{\sigma \dfrac{\dot{C}(t)}{C(t)} + \rho}{\beta L^{1-\beta}} \right)^{\frac{\beta}{\beta - 1}} = \frac{K(t)^\beta}{1 + \beta L^{-\beta} \delta \displaystyle\int_0^t K(s)^\beta \, ds} , \tag{8.21}$$

and we can investigate the behavior of the right hand side of this equation as $t \to \infty$:

$$\lim_{t \to \infty} \frac{K(t)^\beta}{1 + \beta L^{-\beta} \delta \displaystyle\int_0^t K(s)^\beta \, ds} = \lim_{t \to \infty} \frac{K(t)^{\beta - 1} \dot{K}(t)}{\beta L^{-\beta} \delta K(t)^\beta} \quad \text{\color{red}By l'Hôpital's rule}$$

$$= \frac{L^\beta}{\delta} \frac{\dot{K}(t)}{K(t)} .$$

Substituting into (8.21),

$$\sigma \frac{\dot{C}(t)}{C(t)} = \beta \delta^{(1-\beta)/\beta} \left(\frac{\dot{K}(t)}{K(t)} \right)^{(\beta - 1)/\beta} - \rho .$$

It is easy to show that C and K must grow at the same rate along the balanced growth path. So let g denote this common growth rate. Then we get

$$\frac{\sigma g + \rho}{\beta \delta^{(1-\beta)/\beta}} = g^{(\beta - 1)/\beta} ,$$

which implicitly defines g in a manner convenient for graphing (see Figure 8.2). As an exercise, you should show that the comparative statics for this model are the same as for the Lucas-Uzawa model.

There are two observations worth noting here. First, balanced growth paths may only exist for certain parameter values of the model, what is often called in growth theory a knife-edge problem. If the parameter restrictions are not plausible, then perhaps the model is poorly designed. Second, different aggregate growth models yield the same pre-

dictions (e.g. the comparative statics are the same), even though the engine of growth and the policy implications both differ. The learning model has external learning effects implying that the social optimum is to subsidize output. In the Lucas-Uzawa model there is no call to subsidize output at all.   •
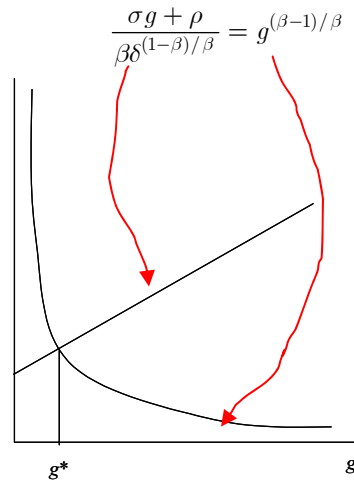
$$\frac{\sigma g + \rho}{\beta \delta^{(1-\beta)/\beta}} = g^{(\beta-1)/\beta}$$

g^* g

FIGURE 8.2

EXERCISE 8.5 (R&D spillovers). *Let* $\alpha_i = q_i / Q_i$ *denote the quality* $q_i(t)$ *of a firm's product relative to the average quality of products* $Q(t)$ *in the industry. Average quality,* $Q(t)$, *grows at the constant exponential rate* β. *Assume the firm's profits (gross of research expenditures) are proportional to relative quality:* $\pi_i(t) = \theta \alpha_i(t)$. *A firm increases its quality* $q_i(t)$ *by spending on R&D. The equation of motion for firm i's quality is* $\dot{q}_i(t) = \gamma q_i(t)^{1-\varepsilon} Q(t)^{\varepsilon} R_i(t)^b$, *where* $\varepsilon < 1$ *and* $b < 1$. *The parameter* ε *measures the degree of spillovers between firms. When* $\varepsilon = 1$ *the rise in firm i's quality depends only on average quality, while when* $\varepsilon = 0$ *it depends only on firm i's quality.*

(a) *Write down the profit maximizing optimal control problem for a firm facing an infinite planning horizon and an interest rate* r. (Hint, express the constraint in terms of relative quality).

(b) *Characterize the evolution of relative quality and R&D.*

(c) *Derive expressions for the steady-state values of R&D and relative quality.*

(d) *What is the speed of convergence to the steady state in the symmetric equilibrium?* (Hint: log-linearize and note that in the symmetric equilibrium $\beta = \gamma\alpha^{-\varepsilon}R^b$). *What are the necessary conditions for stability?*

Further Reading

The standard references on optimal control are Kamien and Schwartz (1981) and Chiang (1992). Both are good, but both have a really annoying flaw: half of each book is taken up with an alternative treatment of dynamic optimization problems called the calculus of variations (CV). CV is a much older technique, less general than optimal control. I suppose CV is in each book because (a) the authors knew it well and (b) you can't fill a whole book with optimal control. Perhaps smaller books at half the price was not option for the publishers.

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